

# THE UNIVERSAL KEPLER PROBLEM

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**ABSTRACT.** For each simple euclidean Jordan algebra, the analogues of hamiltonian, angular momentum and Lenz vector in the Kepler problem are introduced. The analogue of hidden symmetry algebra generated by hamiltonian, angular momentum and Lenz vector is also derived. Finally, for the simplest simple euclidean Jordan algebra, i.e.,  $\mathbb{R}$ , we demonstrate how to get generalized Kepler problems by combining with the quantizations of the TKK algebra.

## 1. INTRODUCTION

Recall that, in the Kepler problem, the hamiltonian is

$$(1.1) \quad H = \frac{1}{2} \mathbf{p}^2 - \frac{1}{r}.$$

Here,  $r$  is length of  $\mathbf{r} \in \mathbb{R}_*^3 := \mathbb{R}^3 \setminus \{\vec{0}\}$  and  $\mathbf{p}$  is the (linear) momentum.

$H$  is clearly invariant under rotations of  $\mathbb{R}^3$ , thanks to Noether's theorem, the angular momentum

$$(1.2) \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

is conserved.

What is special about the Kepler problem is the existence of additional conserved quantity, i.e., the Laplace-Runge-Lenz vector

$$(1.3) \quad \mathbf{A} = \mathbf{L} \times \mathbf{p} + \frac{\mathbf{r}}{r}.$$

It is well-known that  $H$ ,  $\mathbf{L}$  and  $\mathbf{A}$  satisfy the following Poisson bracket relations:

$$(1.4) \quad \left\{ \begin{array}{lcl} \{H, L_i\} & = & 0, \\ \{H, A_i\} & = & 0, \\ \{L_i, L_j\} & = & \epsilon_{ijk} L_k, \\ \{L_i, A_j\} & = & \epsilon_{ijk} A_k, \\ \{A_i, A_j\} & = & -2H \epsilon_{ijk} L_k. \end{array} \right.$$

Here,  $\epsilon_{ijk}$  is the antisymmetric tensor such that  $\epsilon_{123} = 1$ , and a summation over the repeated index  $k$  is assumed in the above.

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When passing to the quantum case, nothing is lost. First of all, we have

$$(1.5) \quad \begin{cases} H &= -\frac{1}{2}\Delta - \frac{1}{r}, \\ \mathbf{L} &= -i\mathbf{r} \times \nabla, \\ \mathbf{A} &= -\frac{i}{2}(\mathbf{L} \times \nabla - \nabla \times \mathbf{L}) + \frac{\mathbf{r}}{r}; \end{cases}$$

secondly, relation (1.4) still holds when Poisson brackets are replaced by commutators.

The goals of this article are to introduce the analogues of  $H$ ,  $\mathbf{L}$  and  $\mathbf{A}$  for each simple euclidean Jordan algebra [1] and then derive the analogue of relation (1.4). To do that, we need to digress into euclidean Jordan algebras and their associated TKK algebras [2].

## 2. TKK ALGEBRAS

Let  $V$  be a simple euclidean Jordan algebra. For  $a \in V$ , we use  $L_a$  to denote the multiplication by  $a$ . Clearly  $L_a$  is an endomorphism on  $V$  and is linearly dependent on  $a$ . We assume the invariant metric  $\langle | \rangle$  on  $V$  is the unique metric such that the unit element  $e$  has unit length, i.e.,

$$\langle a | b \rangle = \frac{1}{\dim V} \text{Tr } L_{ab}.$$

Here, invariant means that  $L_a$  is self-adjoint with respect to  $\langle | \rangle$ , i.e.,  $L'_a = L_a$ .

We denote the Jordan triple product of  $a, b, c$  by  $\{abc\}$ . Recall that

$$\{abc\} = a(bc) - b(ca) + c(ab).$$

We denote the endomorphism  $c \mapsto \{abc\}$  by  $S_{ab}$ . It is clear that

$$S_{ab} = [L_a, L_b] + L_{ab}$$

and is bilinear in  $(a, b)$ . It is clear that  $L_a = S_{ae} = S_{ea}$ . Note that  $S'_{ab} = S_{ba}$ , and if we identify  $V^*$  with  $V$  via the invariant metric,  $S_{ab}^*: V^* \rightarrow V^*$  can be identified with  $-S'_{ba}$ .

One can check that

$$(2.1) \quad [S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{c\{bad\}},$$

so  $S_{ab}$ 's form a real Lie algebra — the **structure algebra**  $\mathfrak{str}$  of  $V$ . The commutation relation in Eq. (2.1) says that, in  $S_{cd}$ ,  $c$  and  $d$  behave as a  $\mathfrak{str}$ -vector and  $\mathfrak{str}$ -covector respectively. In general,  $\mathfrak{str} = \mathfrak{str}' \oplus \mathbb{R}$ , where  $\mathfrak{str}'$ , a semi-simple real Lie algebra, is called the **reduced structure algebra**.

It is an independent discovery of Tits, Kantor, and Koecher [2] that the real reductive Lie algebra  $\mathfrak{str}$  can be naturally extended to a real simple Lie algebra  $\mathfrak{co}$ . As a real vector space, we have

$$\mathfrak{co} = V^* \oplus \mathfrak{str} \oplus V.$$

By writing  $z \in V$  as  $X_z$ ,  $\langle w | \rangle \in V^*$  as  $Y_w$ , the commutation relations can be written as follow: for  $u, v, z, w$  in  $V$ ,

$$(2.2) \quad \boxed{\begin{aligned} [X_u, X_v] &= 0, & [Y_u, Y_v] &= 0, & [X_u, Y_v] &= -2S_{uv}, \\ [S_{uv}, X_z] &= X_{\{uvz\}}, & [S_{uv}, Y_z] &= -Y_{\{vuz\}}, \\ [S_{uv}, S_{zw}] &= S_{\{uvz\}w} - S_{z\{vuw\}}. \end{aligned}}$$

Note that, when the Jordan algebra is the Minkowski space,  $\mathfrak{co} = \mathfrak{so}(2, 4)$  and  $\mathfrak{str} = \mathfrak{so}(1, 3) \oplus \mathbb{R}$ .

By definition, the universal enveloping algebra of  $\mathfrak{co}$  is called the **TKK algebra**. The simply connected Lie group with  $\mathfrak{co}$ , denoted by  $\text{Co}$ , is called the **conformal group**.

### 3. THE UNIVERSAL KEPLER PROBLEM

In the study of generalized Kepler problems [3], we found that, 1) there is a universal relation between the hamiltonian and the generators of the TTK algebra:

$$(3.1) \quad \boxed{H = \frac{1}{2}Y_e^{-1}X_e + iY_e^{-1}},$$

here,  $Y_e$  in the TTK algebra is formally inverted and its formal inverse is denoted by  $Y_e^{-1}$ ; 2) whenever we have a (hidden) geometric realization for a unitary lowest weight representation of  $\text{Co}$ , we get a generalized Kepler problem.

By definition,  $H$  in Eq. (3.1) is called the **universal hamiltonian**, the hypothetical physics problem with  $H$  as its hamiltonian is called the **universal Kepler problem**, and the **universal Lenz vector** is

$$(3.2) \quad \boxed{A_u := iY_e^{-1}[L_u, Y_e^2 H]},$$

i.e.,

$$(3.3) \quad A_u = \frac{i}{2}X_u - iY_u H = \frac{i}{2}(X_u - Y_u Y_e^{-1} X_e) + Y_u Y_e^{-1}.$$

Note that  $A_e = 1$ . One might call  $\vec{A} := (A_{e_1}, \dots, A_{e_{\dim V - 1}})$  the Lenz vector, that is because  $\vec{A}$  is precisely the usual Lenz vector when the Jordan algebra is the Minkowski space.

**Theorem 1** (Main Theorem). *For  $u, v$  in  $V$ , we denote  $[L_u, L_v]$  by  $L_{u,v}$ . Then*

$$(3.4) \quad \boxed{\begin{aligned} [L_{u,v}, H] &= 0, \\ [A_u, H] &= 0, \\ [L_{u,v}, L_{z,w}] &= L_{[L_{u,v}, L_z], w} + L_{z, [L_{u,v}, L_w]}, \\ [L_{u,v}, A_z] &= A_{[L_{u,v}, L_z]}, \\ [A_u, A_v] &= -2H L_{u,v}. \end{aligned}}$$

*Proof.* Eq. (2.2) implies that

$$[L_{u,v}, X_z] = X_{[L_{u,v}, L_z]}, \quad [L_{u,v}, Y_z] = Y_{[L_{u,v}, L_z]}, \quad [L_{u,v}, L_z] = L_{[L_{u,v}, L_z]}.$$

In particular, we have  $[L_{u,v}, X_e] = [L_{u,v}, Y_e] = 0$ . Therefore,  $[L_{u,v}, H] = 0$ ,

$$\begin{aligned} [L_{u,v}, A_z] &= iY_e^{-1}[L_{u,v}, [L_z, Y_e^2 H]] \\ &= iY_e^{-1}([L_{u,v}, L_z], Y_e^2 H) + [L_z, [L_{u,v}, Y_e^2 H]] \\ &= iY_e^{-1}[L_{[L_{u,v}, L_z]}, Y_e^2 H] \\ &= A_{[L_{u,v}, L_z]}, \end{aligned}$$

and

$$\begin{aligned} [L_{u,v}, L_{z,w}] &= [L_{u,v}, [L_z, L_w]] \\ &= [[L_{u,v}, L_z], L_w] + [L_z, [L_{u,v}, L_w]] \\ &= [L_{[L_{u,v}, L_z]}, L_w] + [L_z, L_{[L_{u,v}, L_w]}] \\ &= L_{[L_{u,v}, L_z], w} + L_{z, [L_{u,v}, L_w]}. \end{aligned}$$

Since  $H = Y_e^{-1}(\frac{1}{2}X_e + i)$ , we have

$$\begin{aligned}
[A_u, H] &= [A_u, Y_e^{-1}](\frac{1}{2}X_e + i) + Y_e^{-1}[A_u, \frac{1}{2}X_e + i] \\
&= -Y_e^{-1}[A_u, Y_e]Y_e^{-1}(\frac{1}{2}X_e + i) + Y_e^{-1}[A_u, \frac{1}{2}X_e] \\
&= -Y_e^{-1}[A_u, Y_e]H - Y_e^{-1}\left[\frac{i}{2}X_u - iY_uH, \frac{1}{2}X_e\right] \\
&= -Y_e^{-1}[A_u, Y_e]H - Y_e^{-1}\left[-iY_uY_e^{-1}, \frac{1}{2}X_e\right]Y_eH \\
&= -Y_e^{-1}[A_u, Y_e]H + iY_e^{-1}\left([Y_u, \frac{1}{2}X_e] - Y_uY_e^{-1}[Y_e, \frac{1}{2}X_e]\right)H \\
&= -Y_e^{-1}\left[\frac{i}{2}(X_u - Y_uY_e^{-1}X_e) + Y_uY_e^{-1}, Y_e\right]H + iY_e^{-1}(L_u - Y_uY_e^{-1}L_e)H \\
&= 0.
\end{aligned}$$

Since  $A_u = \frac{i}{2}X_u - iY_uH$ , we have

$$\begin{aligned}
[A_u, A_v] &= [\frac{i}{2}X_u, -iY_vH] - [\frac{i}{2}X_v, -iY_uH] - [Y_uH, Y_vH] \\
&= [\frac{1}{2}X_u, Y_vH] - Y_u[H, Y_v]H - \langle u \leftrightarrow v \rangle \\
&= [\frac{1}{2}X_u, Y_v]H + Y_v[\frac{1}{2}X_u, H] - Y_u[H, Y_v]H - \langle u \leftrightarrow v \rangle \\
&= -S_{uv}H - Y_vY_e^{-1}[\frac{1}{2}X_u, Y_e]H - Y_uY_e^{-1}[\frac{1}{2}X_e, Y_v]H \\
&\quad - \langle u \leftrightarrow v \rangle \\
&= -S_{uv}H + Y_vY_e^{-1}L_uH + Y_uY_e^{-1}L_vH - \langle u \leftrightarrow v \rangle \\
&= -2L_{u,v}H = -2HL_{u,v}.
\end{aligned}$$

□

**3.1. The classical counterpart.** As is well-known, the total cotangent space  $T^*V$  is a natural symplectic space. By virtue of the invariant metric on  $V$ , one can identify  $T^*V$  with  $TV$ , then  $TV$  becomes a symplectic space. The tangent bundle of  $V$  has a natural trivialization, with respect to which, one can denote an element of  $TV$  by  $(x, \pi)$ . We fix an orthonormal basis  $\{e_\alpha\}$  for  $V$  so that we can write  $x = x^\alpha e_\alpha$  and  $\pi = \pi^\alpha e_\alpha$ . Then the basic Poisson bracket relations on  $TV$  are  $\{x^\alpha, \pi^\beta\} = \delta^{\alpha\beta}$ .

Introduce the moment functions

$$(3.5) \quad \mathcal{S}_{uv} := \langle S_{uv}(x) \mid \pi \rangle, \quad \mathcal{X}_u := \langle x \mid \{\pi u \pi\} \rangle, \quad \mathcal{Y}_v := \langle x \mid v \rangle$$

on  $TV$ . Then we have the following easy theorem:

**Theorem 2.** *As polynomial functions on  $TV$ ,  $\mathcal{S}_{uv}$ ,  $\mathcal{X}_u$  and  $\mathcal{Y}_v$  satisfy the following Poisson bracket relations: for any  $u, v, z$  and  $w$  in  $V$ , we have*

$$(3.6) \quad \begin{cases} \{\mathcal{X}_u, \mathcal{X}_v\} = 0, & \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, & \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}, \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, & \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}, \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}. \end{cases}$$

In principle, this theorem should be known to experts on Jordan algebras.

The **classical universal Hamiltonian** and the **classical universal Lenz vector** are

$$(3.7) \quad \boxed{\mathcal{H} = \frac{\frac{1}{2}\mathcal{X}_e - 1}{\mathcal{Y}_e}, \quad \mathcal{A}_u := \frac{\{L_u, \mathcal{Y}_e^2 \mathcal{H}\}}{\mathcal{Y}_e}}$$

respectively. Note that

$$(3.8) \quad \mathcal{A}_u = \frac{1}{2} \left( \mathcal{X}_u - \mathcal{Y}_u \frac{\mathcal{X}_e}{\mathcal{Y}_e} \right) + \frac{\mathcal{Y}_u}{\mathcal{Y}_e}$$

and more explicitly

$$(3.9) \quad \mathcal{H} = \frac{1}{2} \frac{\langle x \mid \pi^2 \rangle}{r} - \frac{1}{r}.$$

Theorem 1 still holds under the following substitutions:

$$[,] \rightarrow \{, \}, \quad H \rightarrow \mathcal{H}, \quad A_u \rightarrow \mathcal{A}_u, \quad L_{u,v} \rightarrow \mathcal{L}_{u,v} := \{\mathcal{L}_u, \mathcal{L}_v\}$$

as one can check directly.

**3.2. A natural question.** Here arises a natural question: how does the universal Kepler problem relate to the  $J$ -Kepler problem in Ref. [3] at the classical level? In order to answer that, we need to make a digression in the next paragraph.

In general, if  $M$  is a manifold,  $N$  is a submanifold of  $M$ , then both  $T^*M$  and  $T^*N$  are symplectic manifolds, however,  $T^*N$  is not a symplectic submanifold of  $T^*M$  because  $T^*N$  is not a submanifold of  $T^*M$  at all. Now if  $M$  is a Riemannian manifold with  $N$  as its Riemannian sub-manifold, then  $T^*N$  can be viewed as a *symplectic* sub-manifold of  $T^*M$  due to the identification between the tangent and cotangent bundle via the Riemannian metric:  $T^*N \cong TN \subset TM \cong T^*M$ . Equivalently, we can say that  $TM$  and  $TN$  are symplectic manifolds because  $M$  and  $N$  are Riemannian, moreover, the fact that  $N$  is a Riemannian submanifold of  $M$  implies that  $TN$  is a symplectic submanifold of  $TM$ . Consequently the Poisson bracket operation commutes with the restriction map from  $C^\infty(TM)$  to  $C^\infty(TN)$ .

For the  $J$ -Kepler problem introduced in Ref. [3], the configuration space is the space  $\mathcal{P}$  consisting of the semi-positive elements of rank one, so it is a submanifold of  $V$ . With the euclidean metric,  $V$  is Riemannian, so  $T\mathcal{P}$  is a symplectic submanifold of  $TV$ . One can check that, by restricting  $\mathcal{H}$  from  $TV$  to  $T\mathcal{P}$ , we obtain the classical Hamiltonian for the  $J$ -Kepler problem, cf. Remark 7.3 in Ref. [3].

How does the universal Kepler problem relate to the  $J$ -Kepler problem in Ref. [3] at the quantum level? This is closely related to the question of quantizing the TKK algebra, which shall be addressed in Ref. [5]. We demonstrate this by working out the simplest example in the next section.

#### 4. THE SIMPLEST EXAMPLE

Here we use the simplest example to demonstrate how to get generalized Kepler problems once we know the quantizations of the TKK algebra.

Throughout this section we assume that the Jordan algebra is  $\mathbb{R}$ . Then the symmetric cone is  $\mathbb{R}_+$  and  $\text{Co} = \widetilde{\text{SL}}(2, \mathbb{R})$  — the universal cover of  $\text{SL}(2, \mathbb{R})$ . A point in  $\mathbb{R}$  is denoted by  $x$ , and the Lebesgue measure on  $\mathbb{R}$  is denoted by  $dx$ . The

conformal algebra is  $\mathfrak{sl}(2, \mathbb{R})$ , with generators  $S := S_{ee}$ ,  $X := X_e$  and  $Y := Y_e$  and commutation relations

$$[S, X] = X, \quad [S, Y] = -Y, \quad [X, Y] = -2S.$$

These generators can be realized as linear differential operators on  $L^2(\mathbb{R}_+, \frac{1}{x} dx)$  as follows:

$$S \rightarrow \tilde{S} := -x \frac{d}{dx}, \quad X \rightarrow \tilde{X}(\nu) := i \left( x \frac{d^2}{dx^2} + \frac{\frac{\nu}{2}(1 - \frac{\nu}{2})}{x} \right), \quad Y \rightarrow \tilde{Y} := -ix.$$

Here,  $\nu$  is a complex parameter whose range is to be determined. Note that, we must specify a common dense domain of definition for these operators. This common domain  $\tilde{D}_\nu$  is defined to be

$$\{x^{\frac{\nu}{2}} e^{-x} p(x) \mid p(x) \in \mathbb{C}[x]\}.$$

For  $\nu \in (0, \infty)$  and only for such an  $\nu$ ,  $x^{\nu-1} e^{-2x} dx$  is a finite positive measure on  $\mathbb{R}_+$ . Therefore, for such and only for such a  $\nu$ ,  $\mathbb{C}[x]$  is dense in  $L^2(\mathbb{R}_+, x^{\nu-1} e^{-2x} dx)$ , or equivalently,  $\tilde{D}_\nu$  is dense in  $L^2(\mathbb{R}_+, \frac{1}{x} dx)$ .

It is not hard to see that operators  $\tilde{S}$ ,  $\tilde{X}(\nu)$  and  $\tilde{Y}$  are all hermitian operators on  $\tilde{D}_\nu$  when  $\nu \in (0, \infty)$ . Therefore,

*for each  $\nu \in (0, \infty)$ ,  $\tilde{D}_\nu$  is a unitary module  $\pi_\nu$  for  $\mathfrak{sl}(2, \mathbb{R})$ ;*

moreover<sup>1</sup>,  $\pi_{\nu_1} \not\cong \pi_{\nu_2}$  if  $\nu_1 \neq \nu_2$ .

Let  $E_\pm = \frac{i}{2}(\tilde{X}(\nu) - \tilde{Y}) \mp \tilde{S}$ ,  $h = \frac{i}{2}(\tilde{X}(\nu) + \tilde{Y})$ . Suppose that  $h(\psi_s) = s\psi_s$  and  $E_-(\psi_s) = 0$ , then  $\psi_s \propto x^s e^{-x}$  with  $s = \nu/2$ . Therefore,  $x^{\frac{\nu}{2}} e^{-x}$  is a lowest weight state for  $\pi_\nu$ . A little play with algebra, one can show that  $\pi_\nu$  is the lowest weight module for  $\mathfrak{sl}(2, \mathbb{R})$ , in fact, a unitary lowest weight  $(\mathfrak{g}, K)$ -modules, where  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ ,  $K \cong \mathbb{R}$  is the simply connected abelian group whose Lie algebra is generated by  $X + Y$ . Since  $\tilde{D}_\nu$  is dense in  $L^2(\mathbb{R}_+, \frac{1}{x} dx)$ , by a theorem of Harish-Chandra, for each  $\nu \in (0, \infty)$ , we have a unitary lowest weight representation (also denoted by  $\pi_\nu$ ) of  $\widetilde{\text{SL}}(2, \mathbb{R})$  on  $L^2(\mathbb{R}_+, \frac{1}{x} dx)$ . In view of the classification theorem for unitary lowest weight modules in Ref. [4], these  $\pi_\nu$  exhaust all nontrivial unitary lowest weight representations of  $\widetilde{\text{SL}}(2, \mathbb{R})$ .

Combining with the result in the previous section, for each  $\nu \in (0, \infty)$ , there is a generalized Kepler problem whose hamiltonian is

$$\tilde{H}(\nu) = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\frac{\nu}{2}(\frac{\nu}{2} - 1)}{2x^2} - \frac{1}{x}.$$

However, the Lenz vector is trivial:  $A_u = u$ . It appears that  $\tilde{H}(\nu) = \tilde{H}(2 - \nu)$  for  $\nu \in (0, 2)$ , but that is not true, because  $\tilde{H}(\nu)$  and  $\tilde{H}(2 - \nu)$  have *different* domains of definition<sup>2</sup>.

<sup>1</sup>It appears that  $\pi_\nu = \pi_{2-\nu}$  for  $\nu \in (0, 2)$ , but that is not true, because  $\tilde{D}_\nu \neq \tilde{D}_{2-\nu}$ .

<sup>2</sup>This can be verified from the requirements that  $\tilde{H}(\nu)$  be hermitian with respect to inner product

$$(f, g) \mapsto \int_{\mathbb{R}_+} \bar{f} g dx,$$

and its domain of definition contain  $x^{\frac{\nu}{2}} e^{-\frac{2x}{\nu}}$ .

The bound state spectrum for  $\tilde{H}(\nu)$  is

$$\left\{ -\frac{1/2}{(I + \nu/2)^2} \mid I = 0, 1, \dots \right\};$$

moreover, being a closed subspace of  $L^2(\mathbb{R}_+, dx)$ , the Hilbert space of bound states for  $\tilde{H}(\nu)$  is isometric to  $L^2\left(\mathbb{R}_+, \frac{1}{x} dx\right)$  via an analogue of the twisting map  $\tau$  introduced in the proof of Theorem 5 in Ref. [3] and hence provides another realization for  $\pi_\nu$ .

In the forthcoming papers, the quantizations of the TKK algebra (hence the generalized Kepler problems) are presented for a generic simple euclidean Jordan algebra.

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